N-Burn Optimal Analytic Trajectories

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N-burn analytic solutions are obtained for propellant-optimal transfer trajectories of a vehicle in a vacuum between arbitrary boundary conditions under the assumptions that the gravitational acceleration vector on the burn subarcs is a linear function of the position vector and that on the coast subarcs the gravitational field is inverse square. Perturbations in the desired boundary conditions are expressed, in general, in terms of perturbations in the control vector and in the initial state vector. All coefficient matrices are computed recursively in terms of the analytic matrices established from the subarcs of the N-burn solution.

Introduction

An N-burn analytic solution can be obtained for the propellant-optimal transfer trajectory of a vehicle in a vacuum between arbitrary boundary conditions if the assumptions are made that the gravitational acceleration vector on the burn subarcs is a linear function of the position vector and that on the coast subarcs the gravitational field is inverse square. With these assumptions, the primer vector on the burn subarcs is represented by a homogeneous, second-order, linear-differential equation that is readily integrated.

The solution of the state equations on the burn subarcs is in terms of two thrust integrals. In Ref. 1, the thrust integrals were evaluated recursively by expanding a portion of the integrand into infinite series. If convergence difficulties were encountered by exceeding either the radius of convergence of the series or the finite word length of the computer, the burn subarcs were simply segmented. However, this procedure increased complexity and solution time. In this report, the two thrust integrals are evaluated more efficiently by use of a characteristic of the integrands; namely, the integrals are smooth, well-behaved functions of time only. In this way, the integrals can be evaluated without recourse to an open-ended series solution.

With this approach, the state and costate vectors at the termination of any subarc can be expressed analytically as functions of the state and costate vectors at the initiation of the subarc and the arc time interval τ. The subarcs are the burn arcs and coast arcs comprising the N-burn solution. The control vector for the problem is composed of the six-component initial costate vector and the switching times for the burn arcs and the coast arcs. By introducing the optimality and transversality conditions for an N-burn solution and by using the homogeneous property of the costate equations, a sufficient number of algebraic equations is obtained in terms of the boundary conditions of the problem and the unknown components of the control vector.

Because analytic expressions exist for all subarcs, a perturbation in the desired boundary conditions can generally be expressed in terms of perturbations in the control vector and perturbations in the initial state vector. Hence, a general analytic algorithm is determined for solving N-burn optimal trajectory problems.

Problem Definition

The optimal control problem can be stated as follows. Minimize the performance index J'

$$J' = \int_{t_I}^{t_F} \frac{T}{\beta} dt \tag{1}$$

subject to the differential constraints

$$\dot{V} = G + (T/m)L \tag{2}$$

$$\dot{R} = V \tag{3}$$

$$\dot{m} = -T/\beta \tag{4}$$

where T is the thrust magnitude, R and V are the position and velocity vectors, G is the gravitational acceleration vector, L is the thrust-direction vector, m is the vehicle mass, and β is the effective exhaust velocity of the engine. The thrust magnitude T is bounded between a minimum value (zero) and a maximum value $T_{\max}$ by the inequality constraint $0 \leqq T \leqq T_{\max}$.

At the initial time t_I, n_I ($n_I \leqq 6$) boundary conditions are imposed on the solution of the form

$$M(R_I, V_I, t_I) = 0 \tag{5}$$

and at the final time t_F, n_F ($n_F \leqq 6$) boundary conditions are imposed of the form

$$N(R_F, V_F, t_F) = 0 \tag{6}$$

In addition, the thrust-direction vector L is required to be a unit vector, or $L^T L = 1$.

The variational Hamiltonian H for this performance index and system of constraint equations is[2]

$$H = T\$ + P^T G + Q^T V \tag{7}$$

where the switch function $\$$ is defined as

$$\$ = 1/\beta - \eta/\beta + p/m \tag{8}$$

P (the primer vector) and Q are the Lagrange multipliers associated with the vectors V and R, η is the time-varying multiplier associated with the mass, and $p = |P|$.

The relationship between the thrust magnitude and the switch function $\$$ can be shown to be[2]

$$T = \begin{cases} T_{\max}, \$ > 0 \\ 0, \$ < 0 \end{cases}$$

The necessary conditions for optimality with respect to the Lagrange multipliers P, Q, and η are given by the relationships

$$\dot{P}^T = -\partial H/\partial V = -Q^T \tag{9}$$

$$\dot{Q}^T = -\partial H/\partial R = -(P^T \nabla) G^T \tag{10}$$

$$\dot{\eta} = -\partial H/\partial m = T p/m^2 \tag{11}$$

where the gravitational acceleration vector G has been assumed to be a function only of position and time.

Burn Arcs

Let the gravitational acceleration vector G on a burn subarc be represented as

$$G_b = -\omega^2 R \tag{12}$$

where the constant ω is equal to the Schuler frequency evaluated

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at the initiation of the burn subarc. The optimizing second-order differential equations representing the position and primer vectors [using Eqs. (2, 3, 9, 10, and 12)] are

$$\ddot{R}+\omega^2 R = (a_0/\mu)P/p \tag{13}$$

$$\ddot{P}+\omega^2 P = 0 \tag{14}$$

where $a_0 = T/m_0$ and $\mu = m/m_0 = 1+\dot{\mu}\tau$.

The differential equation for the primer vector represents in vibration problems the motion of a harmonic oscillator without damping and without a forcing function. The solution for P and $\dot{P}$ is represented by

$$\lambda(\tau) = \psi\lambda(0) \qquad \lambda = \begin{bmatrix} P \\ \dot{P} \end{bmatrix} \tag{15}$$

where ψ is the 6-by-6 matrix

$$\psi = \begin{bmatrix} I\cos\omega\tau & \dfrac{I\sin\omega\tau}{\omega} \\ -I\omega\sin\omega\tau & I\cos\omega\tau \end{bmatrix} \tag{16}$$

I is the 3-by-3 identity matrix, and τ is the subarc time interval.

The second-order differential equation for the position vector [Eq. (13)] has a solution represented by

$$S(\tau) = \psi S(0)+\Omega J \qquad S = \begin{bmatrix} R \\ \dot{R} \end{bmatrix} \tag{17}$$

where the vector J and the matrix Ω are given as follows:

$$J = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \displaystyle\int_0^\tau \frac{P}{\mu p}\cos\omega\tau\, d\tau \\ \displaystyle\int_0^\tau \frac{P}{\mu p}\sin\omega\tau\, d\tau \end{bmatrix} \tag{18}$$

$$\Omega = \frac{a_0}{\omega}\begin{bmatrix} I\sin\omega\tau & -I\cos\omega\tau \\ \omega I\cos\omega\tau & \omega I\sin\omega\tau \end{bmatrix} \tag{19}$$

A method for evaluating the integrals in Eq. (18) is presented in Appendix A.

Coast Arcs

Let the gravitational acceleration vector G on a coast subarc be represented by

$$G_c = -R/r^3 \tag{20}$$

where $r = |R|$. The solution is normalized on the basis of a gravitational constant of unity. For an inverse-square gravitational field, the transformations for the state and costate vectors are known in closed form[3] and may be expressed as

$$S(\tau) = \Upsilon S(0), \quad \lambda(\tau) = \Phi\lambda(0) \tag{21}$$

where Υ and Φ are known 6-by-6 matrices.

A completely closed-form N-burn solution across a series of burn subarcs and coast subarcs can now be computed. For example, if an initial state vector $S(t_I)$ and a control vector consisting of the initial costate vector $\lambda(t_I)$ and the switch times τ_i $(i = 1,\ldots,2N)$ are given, the final state and costate vectors $S(t_F)$ and $\lambda(t_F)$ can be computed by successively using Eqs. (15, 17, and 21).

Perturbation Equations

This initial solution does not necessarily satisfy the desired boundary conditions [Eqs. (5) and (6)]. Therefore, a correction must be made to the control vector so that on the next iteration the solution will be closer to meeting boundary conditions.

The differential of the state and costate vectors at the time $t = \tau_{2i}$ (Fig. 1) for an N-burn solution may be expressed as

$$dU_{2i} = K^{(i)}dU_0 + \sum_{j=1}^{i} A_{i,j}\left(B_{2j-1}\dot{U}_{2j-1}^{-}d\tau_{2j-1} + \sum_{k=1}^{j}\frac{\partial U_{2j}}{\partial \tau_{2k}}d\tau_{2k}\right) \qquad i = 1,2,\ldots,N \tag{22}$$

where

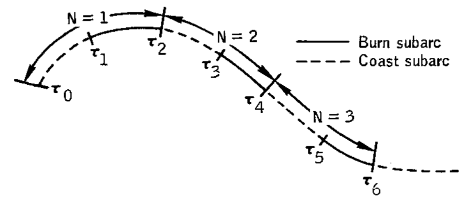

Fig. 1 Definition of arc and subarc sequence.

$$U = \begin{bmatrix} S \\ \lambda \end{bmatrix} \tag{23}$$

$$K^{(i)} = K_i K_{i-1}\cdots K_1 \tag{24}$$

$$K_i = B_{2i-1}C_{2i-2} \tag{25}$$

and

$$B = \begin{bmatrix} \pi & \theta \\ \Xi & \Psi \end{bmatrix} \qquad C = \begin{bmatrix} \Phi & 0 \\ \Gamma & \Phi \end{bmatrix} \tag{26}$$

The matrices π, θ, and Ξ are defined as

$$\pi = \left[\Psi + \left(\frac{\partial\Psi}{\partial\omega}S + \frac{\partial\Omega}{\partial\omega}J + \Omega\frac{\partial J}{\partial\omega}\right)\frac{\partial\omega}{\partial S}\right]$$

$$\theta = \Omega\frac{\partial J}{\partial\lambda} \tag{27}$$

$$\Xi = \frac{\partial\Psi}{\partial\omega}\lambda\frac{\partial\omega}{\partial S}$$

The matrix Γ is defined as $\Gamma = \partial\lambda/\partial S$ and is produced as an additional output from the formulation of the state transition matrix.[2]

The superscript minus in Eq. (22) indicates a quantity evaluated at the time immediately before an event. The subscripts refer to the times at which the variables and matrices are evaluated. Note that N refers to a set of arcs consisting of a coast subarc and a burn subarc. The matrix $\partial J/\partial\lambda$ in Eq. (27) is evaluated in Appendix B.

The reason for the summation within the summation in Eq. (22) is that the vector U at the time τ_{2i} is a function of all the previous burn arc time intervals. This functional dependence occurs through the mass of the vehicle. The vector $\partial U/\partial\tau$ is evaluated in Appendix C.

The 12-by-12 matrix $A_{i,j}$ in Eq. (22) is computed by the following relationships:

$$A_{i,j} = \begin{Bmatrix} 0 & j > i \\ I & j = i \\ K_i & j+1 = i \\ K_i A_{i-1,j} & j+1 < i \end{Bmatrix} \tag{28}$$

where 0 and I are, respectively, the 12-by-12 null matrix and the 12-by-12 identity matrix. For a general gravitational acceleration vector, Eq. (22) can be written only in differential form.[4] The transition matrices for this general case can possibly be evaluated analytically during coast; however, they must be numerically integrated during burns.

In Eq. (22), the differentials in the initial state (as well as in the final state) are included. Hence, general boundary conditions as expressed in Eqs. (5) and (6) can be imposed upon the solution. If the last subarc is a coast, the differentials in the state and costate vectors at the time τ_{2i+1} are

$$dU_{2i+1} = C_{2i}dU_{2i} + \dot{U}_{2i+1}^{-}d\tau_{2i+1} \tag{29}$$

where dU_{2i} is given by Eq. (22).

Additional Optimality and Transversality Conditions

Let F_i $(i = 1,\ldots,6)$ represent the difference between the desired final state and the final state actually computed. For boundary conditions of this type, F_i $(i = 1,\dots,6)$ represents six equations in terms of $2N+6$ unknowns $[\lambda_0, \tau_i(i = 1,\dots,2N)]$. The additional $2N$ constraint equations are obtained from the following conditions: 1) the optimality and transversality conditions that must be satisfied on an optimum N-burn trajectory, and 2) the homogeneous property of the costate equations.

If the gravitational acceleration vector is expressed as the function

$$G = [TG_b + (T_{max} - T)G_c]/T_{max} \tag{30}$$

the modified switch function is

$$\$ = 1/\beta - \eta/\beta + p/m + P^T(G_b - G_c)/T_{max} \tag{31}$$

At the times τ_{2i} and τ_{2i+1} $(i = 1,\dots,N-1)$, this switch function must necessarily be equal to zero (see conditions between the thrust magnitude T and the switch function $). By evaluating the switch function at these pairs of times [while noting that the mass m and its multiplier η from Eq. (11) are constant over the interval] the following $N-1$ conditions result

$$F_{i+6} = (\beta_{2i+1}p_{2i+1} - \beta_{2i}p_{2i}) - \mu_{2i}\beta_{2i}\frac{(P_{2i}{}^T \Delta G_{2i})}{T_{2i}/m_0} \qquad i = 1,\dots,N-1 \tag{32}$$

where different exhaust velocities β have been assumed on subsequent burn arcs and the discontinuity in the gravitational model $\Delta G = (G_b - G_c)$ at the junctions of the subarcs has been taken into account.

The total differential of Eq. (32) is

$$dF_{i+6} = \sigma_{2i}{}^T dU_{2i} + \mathcal{K}_{2i+1} d\tau_{2i+1} + \mathcal{F}_{2i}\sum_{j=1}^{i} \dot{\mu}_{2j} d\tau_{2j} + \mathcal{M}_{2i-1} d\tau_{2i-1} + \mathcal{N}_{2i-2} dS_{2i-2} \qquad i = 1,2,\dots,N-1 \tag{33}$$

where

$$\sigma_{2i} = \begin{bmatrix} \mathcal{I} \\ \mathcal{J} \end{bmatrix}_{2i}$$

and

$$\mathcal{I}_{2i}{}^T = \beta_{2i+1} 1_{2i+1}{}^T \Gamma_{2i} + \mathcal{E}_{2i}{}^T$$

$$\mathcal{J}_{2i}{}^T = \beta_{2i+1} 1_{2i+1}{}^T \Phi_{2i} - \beta_{2i} 1_{2i}{}^T - \mathcal{G}_{2i}{}^T$$

$$\mathcal{K}_{2i+1} = \beta_{2i+1} 1_{2i+1}{}^T \dot{\lambda}_{2i+1}{}^-$$

$$\mathcal{M}_{2i-1} = -\mathcal{D}_{2i-1}{}^T \dot{S}_{2i-1}{}^-$$

$$\mathcal{N}_{2i-2} = -\mathcal{D}_{2i-1}{}^T \Phi_{2i-2}$$

$$\mathcal{F}_{2i} = -\beta_{2i}\frac{(P_{2i}{}^T \Delta G_{2i})}{T_{2i}/m_0}$$

The 1-by-6 vectors $1_{2i}{}^T$ and $1_{2i+1}{}^T$ are

$$1_{2i}{}^T = \begin{bmatrix} \dfrac{P_{2i}{}^T}{p_{2i}} & \vdots & 0 \end{bmatrix}$$

$$1_{2i+1}{}^T = \begin{bmatrix} \dfrac{P_{2i+1}{}^T}{p_{2i+1}} & \vdots & 0 \end{bmatrix}$$

and the 1-by-6 vectors $\mathcal{D}^T$, $\mathcal{E}^T$, and $\mathcal{G}^T$ are

$$\mathcal{D}_{2i-1}{}^T = \left[\frac{\mu_{2i}\beta_{2i}P_{2i}{}^T}{T_{2i}}\left(\frac{3R_{2i}R_{2i-1}{}^T}{|R_{2i-1}|^5}\right) : 0\right]$$

$$\mathcal{E}_{2i}{}^T = \left\{\frac{\mu_{2i}\beta_{2i}P_{2i}{}^T}{T_{2i}}\left[\frac{3R_{2i}R_{2i}{}^T}{|R_{2i}|^5} - \left(\frac{1}{|R_{2i}|^3} - \omega^2\right)I\right] : 0\right\}$$

$$\mathcal{G}_{2i}{}^T = \left[\frac{\mu_{2i}\beta_{2i}\Delta G_{2i}{}^T}{T_{2i}} : 0\right]$$

Another set of conditions that must be satisfied if the switch times are to take on optimum values is that the Hamiltonian must be constant across the solution. At the times τ_{2i} and τ_{2i-1}, the difference in the Hamiltonians may be used as an additional N constraint equations

$$F_{i+N+5} = H_{2i} - H_{2i-1} \qquad i = 1,\dots,N \tag{34}$$

The total differential of Eq. (34) is

$$dF_{i+N+5} = \gamma_{2i}{}^T dU_{2i} - \gamma_{2i-1}{}^T dU_{2i-1} \tag{35}$$

where

$$\gamma_{2i} = \begin{bmatrix} \dot{p}^+ \\ -\dot{P} \\ \dot{V}^+ \\ -V \end{bmatrix}_{2i}$$

$$\gamma_{2i-1} = \begin{bmatrix} \dot{p}^- \\ -\dot{P} \\ \dot{V}^- \\ -V \end{bmatrix}_{2i-1} \qquad i = 1,2,\dots,N$$

The $(2N+6)$th constraint equation is obtained from the condition that the costate vector is represented by a homogeneous set of equations or that not all of the components of λ_0 are independent. The $(2N+6)$th constraint equation is written as

$$F_{2N+6} = \lambda_0{}^T\lambda_0 - \text{const} \tag{36}$$

where the constant is computed to be the inner product of the input costate vector.

The total differential of Eq. (36) is simply

$$dF_{2N+6} = 2\lambda_0{}^T d\lambda_0 \tag{37}$$

The system of constraint equations F_i $(i = 1,\dots,2N+6)$ can be solved for the control vector using the perturbation equations [Eqs. (22, 33, 35, and 37)] by a near-quadratic convergence technique.[5] This mathematical algorithm has characteristics such that in the first few iterations it behaves like a gradient technique and as the convergence progresses, if shifts to a Newton method.

Appendix A: Evaluation of Thrust Integrals

To evaluate the integrals I_1 and I_2 of Eq. (18), note that the integrands are known functions of time only. Thus, if the burn-arc time τ is known, the integrands, or $\dot{I}_1$ and $\dot{I}_2$, will be known for every time within the burn-arc time interval $(0, \tau)$. Note also that the integrands are smooth, well-behaved functions of time; hence, it is reasonable to assume that I_1 and I_2 may be expressed as a power series in time

$$I_i = \sum_{j=1}^{n} a_{i,j}\tau^{j-1} \qquad i = 1,2 \tag{A1}$$

If the assumption is made that the differential of the function I_i can be represented by the differential of its power series expansion,[6] then $\dot{I}_1$ can be expressed as

$$\dot{I}_i = \sum_{j=2}^{n} (j-1)a_{i,j}\tau^{j-2} \tag{A2}$$

Values for the coefficients $a_{i,j}$ $(i = 1,2; j = 1,\dots,n)$ are obtained by evaluating the integrals and their derivatives at selected times within the burn-arc time intervals $(0, \tau)$. Because the integrands are smooth functions, n will arbitrarily equal 4. At the time $\tau = 0$, Eqs. (A1) and (A2) are

$$I_i(0) = a_{i,1}$$

$$\dot{I}_i(0) = a_{i,2}$$

At the times $\tau/2$ and τ, Eq. (A2) can be solved for $a_{i,3}$ and $a_{i,4}$

$$a_{i,3} = \frac{[2\dot{I}_i(\tau/2) - \dot{I}_i(\tau)/2 - 3\dot{I}_i(0)/2]}{\tau}$$

$$a_{i,4} = \frac{2[\dot{I}_i(\tau) - 2\dot{I}_i(\tau/2) + \dot{I}_i(0)]}{3\tau^2}$$

Using the coefficients $a_{i,j}$ in Eq. (A1), the integrals I_i are

$$I_i = I_i(0) + [\dot{I}_i(0) + 4\dot{I}_i(\tau/2) + \dot{I}_i(\tau)]\,\tau/6 \qquad i = 1,2 \tag{A3}$$

Appendix B: Evaluation of $\partial J/\partial\lambda$

From Appendix A, the partial derivative of the integrals I_1 and I_2 with respect to λ may be written as

$$\frac{\partial I_i}{\partial \lambda_0} = \left[\frac{\partial \dot{I}_i(0)}{\partial \lambda_0} + \frac{4\partial \dot{I}_i(\tau/2)}{\partial \lambda_0} + \frac{\partial \dot{I}_i(\tau)}{\partial \lambda_0}\right]\frac{\tau}{6} \qquad i = 1,2 \tag{B1}$$

Thus, obtaining the partial derivatives of the integrals I_i across a burn subarc requires only evaluating the partial derivative of $\dot{I}_i$ at specific times within the burn subarc time intervals. Also, because

$$\partial\dot{I}_2/\partial\lambda = (\partial\dot{I}_1/\partial\lambda)\tan\omega\tau \tag{B2}$$

[Eq. (18)], it is necessary only to evaluate the partial derivative of one of these integrals. Using Eq. (15), $\dot{I}_1$ may be expressed as

$$\dot{I}_1 = [\dot{P}_0\sin\omega\tau + \omega P_0\cos\omega\tau]\cos\omega\tau/\omega p\mu \tag{B3}$$

This equation can be expressed in the following form

$$\dot{I}_1 = (\cos\omega\tau/\mu p)\Lambda\lambda_0 \tag{B4}$$

where the 3-by-6 matrix Λ and the vector λ_0 are

$$\Lambda = \left[I\cos\omega\tau \; \vdots \; I\frac{\sin\omega\tau}{\omega} \right], \qquad \lambda_0{}^T = [P_0{}^T : \dot{P}_0{}^T] \tag{B5}$$

and I is the 3-by-3 identity matrix.

The differential of Eq. (B4) is

$$d\dot{I}_1 = \frac{\cos\omega\tau}{\mu}\left[\frac{\Lambda\, d\lambda_0}{p} - \frac{(P^T dP)\Lambda\lambda_0}{p^3}\right] + \ldots \tag{B6}$$

From Eq. (B4), note that the vector P is expressed as

$$P = \Lambda\lambda_0 \tag{B7}$$

Using Eq. (B7) and the differential of Eq. (B7) in Eq. (B6), $d\dot{I}_1$ is

$$d\dot{I}_1 = \frac{\cos\omega\tau}{\mu}\left\{\frac{\Lambda\, d\lambda_0}{p} - \frac{\Lambda\lambda_0}{p^3}[(\Lambda\lambda_0)^T(\Lambda\, d\lambda_0)]\right\} + \ldots \tag{B8}$$

By factoring the term Λ/p to the left and the term $d\lambda_0$ to the right, Eq. (B8) can be written as

$$d\dot{I}_1 = K\, d\lambda_0 + \ldots \tag{B9}$$

where the 3-by-6 matrix K is

$$K = (\Lambda/\mu p)[I - \lambda_0(\Lambda\lambda_0)^T\Lambda/p^2]\cos\omega\tau \tag{B10}$$

and I is a 6-by-6 identity matrix. Thus, the partial derivative of the integrals I_1 and I_2 with respect to the vector λ_0 are

$$\begin{aligned} \partial I_1/\partial\lambda_0 &= [K(0) + 4K(\tau/2) + K(\tau)]\,\tau/6 \\ \partial I_2/\partial\lambda_0 &= [4K(\tau/2)\tan(\omega\tau/2) + K(\tau)\tan\omega\tau]\,\tau/6 \end{aligned} \tag{B11}$$

Appendix C: Evaluation of $\partial U/\partial\tau$

From Eqs. (15) and (17), the partial derivatives of the vector U with respect to the arc time interval τ is

$$\frac{\partial U}{\partial\tau} = \begin{bmatrix} \dfrac{\partial S}{\partial\tau} \\ \dfrac{\partial\lambda}{\partial\tau} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial\Psi}{\partial\tau}S(0) + \dfrac{\partial\Omega}{\partial\tau}J + \Omega\dfrac{\partial J}{\partial\tau} \\ \dfrac{\partial\psi}{\partial\tau}\lambda(0) \end{bmatrix} \tag{C1}$$

Since the integrals I_1 and I_2 of Eq. (18) were evaluated approximately, the vector $\partial S/\partial\tau$ is not equal to V and $\dot{V}$ evaluated at the time τ. However, the vector $\partial\lambda/\partial\tau$ is equal to $\dot{P}$ and $\ddot{P}$ evaluated at the time τ since the differential equation for P [Eq. (14)] was solved exactly. Hence, only the vector $\partial S/\partial\tau$ (and specifically the term $\partial J/\partial\tau$) requires further definition. Defining the burn arc time intervals as $\tau_1, \tau_2, \ldots, \tau_{i-1}, \tau_i$, the vector dJ on the ith burn subarc may be expressed as

$$dJ = \frac{dJ}{\partial\tau_i}d\tau_i + \sum_{j=1}^{i-1}\frac{\partial J}{\partial\mu_i}\frac{\partial\mu_i}{\partial\tau_j}d\tau_j + \ldots \tag{C2}$$

The term $\partial\mu_i/\partial\tau_j$ is simply $\dot{\mu}_j$ (the mass flow rate) on the jth burn arc. From Appendix A, the vectors I_1 and I_2 are

$$\left.\begin{aligned} I_1 &= (\tau/6)[\dot{I}_1(0) + 4\dot{I}_1(\tau/2) + \dot{I}_1(\tau)] \\ I_2 &= (\tau/6)[4\dot{I}_1(\tau/2)\tan(\omega\tau/2) + \dot{I}_1(\tau)\tan\omega\tau] \end{aligned}\right\} \tag{C3}$$

Since

$$J = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

the first term in Eq. (C2) is

$$\begin{aligned} \frac{\partial I_1}{\partial\tau} &= \frac{I_1}{\tau} + \frac{\tau}{6}\left[4\frac{\partial\dot{I}_1(\tau/2)}{\partial\tau} + \frac{\partial\dot{I}_1(\tau)}{\partial\tau}\right] \\ \frac{\partial I_2}{\partial\tau} &= \frac{I_2}{\tau} + \frac{\tau}{6}\left[4\frac{\partial\dot{I}_1(\tau/2)}{\partial\tau}\tan\frac{\omega\tau}{2} + \frac{\partial\dot{I}_1(\tau)}{\partial\tau}\tan\omega\tau + 2\omega\dot{I}_1\left(\frac{\tau}{2}\right)\sec^2\frac{\omega\tau}{2} + \omega\dot{I}_1(\tau)\sec^2\omega\tau\right] \end{aligned} \tag{C4}$$

The terms $\partial\dot{I}_1/\partial\tau$ are

$$\begin{aligned} \frac{\partial\dot{I}_1(\tau/2)}{\partial\tau} &= \frac{1}{2}\left[\dot{P}\left(\frac{\tau}{2}\right) - W\left(\frac{\tau}{2}\right)P\left(\frac{\tau}{2}\right)\right]\frac{\cos\omega\tau/2}{\mu(\tau/2)p(\tau/2)} \\ \frac{\partial\dot{I}_1(\tau)}{\partial\tau} &= [\dot{P}(\tau) - W(\tau)P(\tau)]\frac{\cos\omega\tau}{\mu(\tau)p(\tau)} \end{aligned} \tag{C5}$$

where

$$p = |P| \tag{C6}$$

and

$$W(\tau) = \omega\tan\omega\tau + \frac{\dot{\mu}(\tau)}{\mu(\tau)} + \frac{P^T(\tau)\dot{P}(\tau)}{p^2(\tau)} \tag{C7}$$

The terms $\partial J/\partial\mu_j$ are

$$\begin{aligned} \frac{\partial I_1}{\partial\mu} &= \frac{\tau}{6}\left[\frac{\partial\dot{I}_1(0)}{\partial\mu} + \frac{4\partial\dot{I}_1(\tau/2)}{\partial\mu} + \frac{\partial\dot{I}_1(\tau)}{\partial\mu}\right] \\ \frac{\partial I_2}{\partial\mu} &= \frac{\tau}{6}\left[4\left(\tan\frac{\omega\tau}{2}\right)\frac{\partial\dot{I}_1(\tau/2)}{\partial\mu} + (\tan\omega\tau)\frac{\partial\dot{I}_1(\tau)}{\partial\mu}\right] \end{aligned} \tag{C8}$$

where

$$\partial\dot{I}_1(\tau)/\partial\mu = -P(\tau)\cos\omega\tau/\mu^2(\tau)p(\tau) \tag{C9}$$

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